Math 2010 Supplementary Notes   Prof. Y. Kwong

More Examples on limits using ε & δ:

Definition Let $f: D \to \mathbb{R}$ where D is a domain in $\mathbb{R}^n$, then $\forall \vec{a} \in D$,

$$\lim_{\vec{x} \to \vec{a} \in D} f(\vec{x}) = L \text{ iff } \forall \varepsilon > 0, \exists \delta = \delta(a, \varepsilon) > 0$$

s.t. whenever $0 < |\vec{x} - \vec{a}| < \delta$, $|f(\vec{x}) - L| < \varepsilon$.

Ex Show that $\lim_{(x,y) \to (1,2)} f(x,y) = 13$ where $f(x,y) = 3x + 5y$

Pf:

$(x-1+1)$ $(y-2+2)$

$$|f(x,y) - 13| = |3x + 5y - 13| = |3(x-1) + 5(y-2)|$$

$$\leq 3|x-1| + 5|y-2| \leq 8\sqrt{(x-1)^2 + (y-2)^2}$$

$$= 8|(x,y) - (1,2)| \quad \text{here } \vec{x} = \langle x, y\rangle,\ \vec{a} = \langle 1, 2\rangle$$

In essence, $|f(x,y) - 13|$ could be controlled by $|(x,y) - (1,2)|$ i.e. the distance between (x,y) and $(1,2)$.

Now that $8|(x,y) - (1,2)| \leq 8\delta$ if $|(x,y) - (1,2)| < \delta$ implying $|f(x,y) - 13| \leq 8\delta$. Therefore, inorder to have $|f(x,y) - 13| \leq \varepsilon$, it suffices to pick δ s.t. $8\delta = \varepsilon \Rightarrow \delta = \varepsilon/8$ //.

Ex Given $f: \mathbb{R}^2 \to \mathbb{R}$, with $f(x,y) = xy$

Prove that $\lim_{(x,y) \to (2,3)} f(x,y) = 6$

Pf:

Once more, the key point would be trying to control or to dominate $|f(x,y) - 6|$ by $|(x,y) - (2,3)|$

Math 2010 Supplementary Notes Prof. Y. Kwong

$$|f(x,y)-6| = |xy-6|$$

$$= |(x-2+2)(y-3+3)-6| = |(x-2)(y-3)+(x-2)3+2(y-3)|$$

$$\leq |x-2||y-3| + 3|x-2| + 2|y-3|$$

Now that if $|(x,y)-(2,3)| = \sqrt{(x-2)^2+(y-3)^2} < \delta$,

We have $|f(x,y)-6| \leq \delta^2 + 5\delta$

We now take a preliminary estimate on δ, we could restrict δ to be $\delta \in (0,1]$ or $0<\delta\leq 1$. Then

$$|f(x,y)-6| \leq \delta + 5\delta = 6\delta.$$

Therefore, if we desire to have $|f(x,y)-6| < \varepsilon$, we could pick $6\delta = \varepsilon$ so that $\delta = \varepsilon/6$

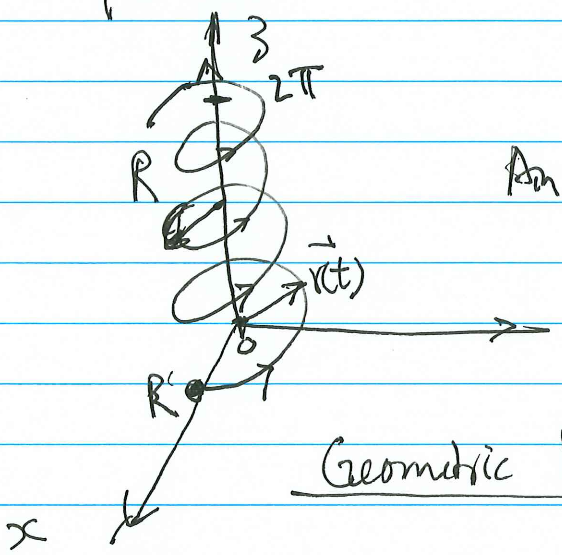
However, we begin with the preliminary estimate on δ i.e. $0<\delta\leq 1$, it therefore suffices to pick $\delta = \min\{1, \varepsilon/6\}$ //.

Vector-valued functions

Motivating Examples

Ex 1. Space Curves

$\vec{r}(t) \,\big|\, [0, 2\pi] \longrightarrow \mathbb{R}^3$ e.g. $\vec{r}(t) = \langle R\cos t, R\sin t, t\rangle$ $t \in (0, \infty)$ for some $R > 0$.



An Spiral or a helix.

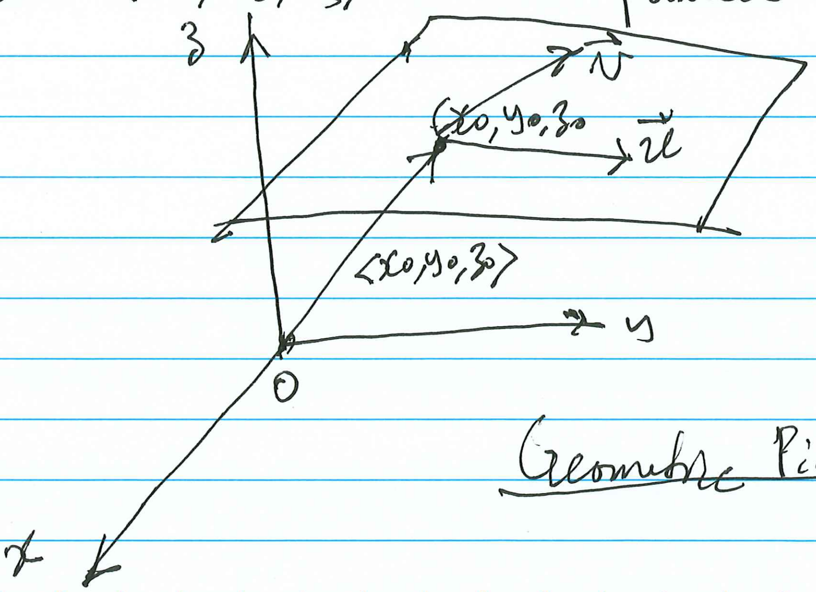
Geometric Pictures

Ex 2. Planes in Space

$$\vec{r}(u, v) = \langle x_0, y_0, z_0\rangle + s\underbrace{\langle u_1, u_2, u_3\rangle}_{\vec{u}} + t\underbrace{\langle v_1, v_2, v_3\rangle}_{\vec{v}}$$

$$\text{for } (s, t) \in \mathbb{R}^2 = (-\infty, \infty) \times (-\infty, \infty)$$

where (x_0, y_0, z_0) is a specific point in space, $\langle u_1, u_2, u_3\rangle$ and $\langle v_1, v_2, v_3\rangle$ are non-parallel vectors.



$\vec{r} \,\big|\, \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

Geometric Picture

Math 2010 Supplementary Notes Prof. Y. Kwong

In general, a vector-valued function is one which takes the form:

$$\vec{f} \,\Big|\, D \subseteq \mathbb{R}^n \to \mathbb{R}^m \qquad n, m \geq 1$$

where D is a domain in $\mathbb{R}^n$ (i.e. D is an open connected set in $\mathbb{R}^n$).

$\vec{f}$ takes the form

$$\vec{f}(\vec{x}) = \langle f_1(\vec{x}), f_2(\vec{x}), \cdots, f_m(\vec{x}) \rangle \qquad \vec{x} \in D$$

Or, by setting $\vec{x} = \langle x_1, x_2, \cdots, x_n \rangle \in D$

$$\vec{f}(x_1, \cdots x_n) = \langle f_1(x_1, \cdots, x_n), f_2(x_1, \cdots, x_n), \cdots f_m(x_1, \cdots, x_n) \rangle$$

where $f_1(\vec{x}), \cdots, f_m(\vec{x})$ $\vec{x} \in D$ are known as the component functions of $\vec{f}$.

Matrix Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \updownarrow m$$
$$\longleftarrow n \longrightarrow$$

is known as a $m \times n$ matrix i.e. there are m rows and n columns of entries. We let a_{ij} $1 \leq i \leq m$, $1 \leq j \leq n$

denote the entry at the junction of the i^{th} row and the jth column

We could also express it as,

$$A = \begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_m \end{bmatrix}$$

where $\vec{a}_1 = \langle a_{11}, a_{12}, \cdots, a_{1n} \rangle$

$\vec{a}_2 = \langle a_{21}, a_{22}, \cdots a_{2n} \rangle$

$\vdots$

$\vec{a}_m = \langle a_{m1}, a_{m2}, \cdots, a_{mn} \rangle$

they are m row vectors in $\mathbb{R}^n$.

Math 2010 Supplementary Notes. Prof. Y. Kwong

Similarly, suppose we have a $n \times k$ matrix B i.e.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots \\ b_{n1}, & b_{n,2}, & \cdots, & b_{n,k} \end{bmatrix} = \begin{bmatrix} \vec{b_1} \cdot \vec{b_2} \cdots \vec{b_k} \end{bmatrix}$$

where $\vec{b_1} = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix}$, $\vec{b_2} = \begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{n2} \end{bmatrix}$, $\cdots$, $\vec{b_k} = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix} \in \mathbb{R}^n$

There are k column vectors in $\mathbb{R}^n$, each with n components.

Consider now $\vec{a_i}$, $1 \le i \le m$ is one of the m row vectors from A and $\vec{b_j}$, $1 \le j \le k$ is one of the k column vectors in B.

Then $\vec{a_i} \cdot \vec{b_j}$ is naturally given by

dot product

$$\vec{a_i} \cdot \vec{b_j} = [a_{i1}, a_{i2}, \cdots, a_{in}] \cdot \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$ Note that each of them got n components

$$= a_{i1} \cdot a_{1j} + a_{i2} \cdot a_{2j} + \cdots + a_{in} \cdot b_{nj}$$ ← i.e. could be multiplied component wise as usual.

Now that we a ready to define the product

$$\underset{m \times n}{A} \; \underset{n \times k}{B} = C_{m \times k}$$ ← Their product which we denote by C would be a $m \times k$ matrix

$$= \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{bmatrix}$$ where c_{ij} $1 \le i \le m$ & $1 \le j \le k$

is given by $c_{ij} = \vec{a_i} \cdot \vec{b_j}$ i^{th} row of $A \cdot j^{th}$ column of B

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Remark: Note that $\vec{a}_i$ and $\vec{b}_j$ both have n components, their dot product is well defined. This explains in order to have the multiplication between AB to be well defined, we must have

$$\underset{m\times k}{C} = \underset{m\times n}{A}\ \underset{n\times k}{B}$$

where the no. of columns of A must match the no. of rows of B.

Ex.

$$\underset{2\times 2}{\overset{A}{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}}\ \underset{2\times 3}{\overset{B}{\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}}} = \underset{2\times 3}{\overset{C}{\begin{bmatrix} 21 & 24 & 27 \\ 47 & 59 & 61 \end{bmatrix}}}$$

$$c_{21} = \vec{a}_2 \cdot \vec{b}_1 = [3, 4]\begin{bmatrix} 5 \\ 8 \end{bmatrix} = 15+32=47$$

Once we are equipped with matrix multiplications, we are ready to go back to our vector-valued functions.

$$\vec{f} \mid D \subseteq \mathbb{R}^n \to \mathbb{R}^m$$

with

$$\vec{f}(\vec{x}) = \langle f_1(\vec{x}), \cdots, f_m(\vec{x})\rangle \text{ or } \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}, \quad \vec{x} \in D.$$

Let us assume that $\frac{\partial f_i(\vec{a})}{\partial x_j}$ exists $\forall\ 1 \le i \le m,\ 1 \le j \le n$, where $\vec{a} \in D \subseteq \mathbb{R}^n$ and $\frac{\partial f_i}{\partial x_j}$ are continuous as well.

Corresponding to $f_i(\vec{x})$, $1 \le i \le m$, it is a function of n variables. By our theory on functions of n variables, we have from pp. 83–85

$$\underset{1\times 1}{f_i(\vec{x})} = \underset{1\times 1}{f_i(\vec{a})} + \underbrace{\underset{1\times n}{\nabla f_i(\vec{a})} \cdot \underset{n\times 1}{(\vec{x}-\vec{a})}}_{1\times 1} + \underbrace{\underset{1\times n}{\overset{1\times 1}{\vec{\varepsilon}_i(|\vec{x}-\vec{a}|)}} \cdot \underset{n\times 1}{(\vec{x}-\vec{a})}}_{\delta_i(|\vec{x}-\vec{a}|) = o(|\vec{x}-\vec{a}|)}$$

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Here $\nabla f_i(\vec{a}) = \underbrace{\left\langle \frac{\partial f_i}{\partial x_1}(\vec{a}), \dots, \frac{\partial f}{\partial x_n}(\vec{a}) \right\rangle}_{1\times n}$ $\quad \vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in D \subseteq \mathbb{R}^n$

$\vec{x} - \vec{a} = \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}$ $n\times 1$ $\quad \vec{\varepsilon}_i(|\vec{x}-\vec{a}|) = \langle \varepsilon_{i1}(|\vec{x}-\vec{a}|, \dots \varepsilon_{in}(|\vec{x}-\vec{a}|) \rangle$

corresponding to f_i $\qquad$ 0 as $\vec{x}\to\vec{a}$

$\delta_i = \vec{\varepsilon}_i \cdot (\vec{x}-\vec{a}) = o(|\vec{x}-\vec{a}|)$

Alternatively we could also expressed

$$f_i(\vec{x}) = f_i(\vec{a}) + \nabla f_i(\vec{a}) \cdot (\vec{x}-\vec{a}) + o(|\vec{x}-\vec{a}|) \quad 1 \le i \le m$$

$\nabla f_i(\vec{a})$ is actually the total derivative of f_i at the pt. $\vec{a}$.

Putting all the equations of $f_i(x)$ together for $i = 1, \dots, m$,

$$f_1(\vec{x}) = \overbrace{f_1(\vec{a}) + \nabla f_1(\vec{a}) \cdot (\vec{x}-\vec{a})}^{L_1(\vec{x})} + \delta_1(|\vec{x}-\vec{a}|)$$

(c.f. bottom of page 6)

$$f_2(\vec{x}) = \overbrace{f_2(\vec{a}) + \nabla f_2(\vec{a}) \cdot (\vec{x}-\vec{a})}^{L_2(\vec{x})} + \delta_2(|\vec{x}-\vec{a}|)$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$f_m(\vec{x}) = f_m(\vec{a}) + \nabla f_m(\vec{a}) \cdot (\vec{x}-\vec{a}) + \delta_m(|\vec{x}-\vec{a}|)$$

error terms

Summing them up together,

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(\vec{a}) \\ \vdots \\ f_m(\vec{a}) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f}{\partial x_n}(\vec{a}) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}}_{m\times n \text{ matrix}} \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_m - a_n \end{bmatrix} + \overbrace{\begin{bmatrix} \delta_1(\vec{x}-\vec{a}) \\ \vdots \\ \delta_m(\vec{x}-\vec{a}) \end{bmatrix}}^{\text{Errors}}$$

all $o(|\vec{x}-\vec{a}|)$

$$[J_f]_{m\times n}(\vec{a}) = \left[\frac{\partial f_i(\vec{a})}{\partial x_j} \right]_{\substack{1\le i \le m \\ 1 \le j \le n}}$$

is known as the Jacobian Matrix

Math 2010 Supplementary Notes Prof. Y. Kwong

$$\vec{f}(\vec{x}) = \vec{f}(\vec{a}) + [J_f]_{m\times n}(\vec{a})\cdot(\vec{x}-\vec{a}) + \vec{\delta}(|\vec{x}-\vec{a}|)$$

where $\boxed{\lim_{\vec{x}\to\vec{a}} \dfrac{|\vec{\delta}(|\vec{x}-\vec{a}|)|}{|\vec{x}-\vec{a}|} = 0}$

Denote $\vec{L}(\vec{x}) = \vec{f}(\vec{a}) + \underset{m\times n}{[J_f]}(a)\cdot\underset{n\times 1}{(\vec{x}-\vec{a})}$ which is the linear approximation of f at $\vec{a}$.

We have $\vec{f}(\vec{x}) = \vec{L}(x) + \vec{\delta}(|x-a|)$

error term when approximate $\vec{f}(\vec{x})$ by $\vec{L}(\vec{x})$

or $\vec{\delta}(|\vec{x}-\vec{a}|) = \vec{f}(\vec{x}) - \vec{L}(\vec{x})$

c.f. assignment 5

Remarks:

(i) $J_f(\vec{a})$ corresponds to the total derivative of $\vec{f}$ at $\vec{a}$.

(ii) The differential of $\vec{f}$ at $\vec{a}$ is given by

$$\vec{df}(a) = J_f(a)\cdot d\vec{x} \quad \text{where } d\vec{x} = \begin{bmatrix} dx_1 \\ \vdots \\ dx_n \end{bmatrix} = \vec{\Delta x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix}$$

$$\vec{\Delta f} = \begin{bmatrix} \Delta f_1 \\ \vdots \\ \Delta f_m \end{bmatrix}$$

and

$$\vec{\Delta f} \simeq \vec{df}$$ i.e. the differential of $\vec{f}$ is regarded as an approximation for $\vec{\Delta f}$.

Math 2010 Supplementary Notes Prof. Y. Kwong

Ex $\vec{f}: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\vec{f}(x,y) = \langle \underbrace{(y+1)\ln x}_{f_1}, \underbrace{x^2 - \sin y + 1}_{f_2} \rangle$

$$= \begin{bmatrix} (y+1)\ln x \\ x^2 - \sin y + 1 \end{bmatrix}$$ (rewriting as a column vector)

Compute $J_{\vec{f}}(1,0) = \begin{bmatrix} \frac{y+1}{x} & \ln x \\ 2x & -\cos y \end{bmatrix}_{\substack{x=1 \\ y=0}} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ total derivative of $\vec{f}$ at $\vec{a} = (1,0)$

Suppose now we want to find an approximate value for $\vec{f}(0.9, 0.1)$. using $(x,y) = (1,0)$ as a reference pt.

from $(1,0)$ to $(0.9, 0.1)$ $d\vec{x} = \Delta\vec{x} = \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$

Now that $\Delta\vec{f} = \vec{f}(0.9, 0.1) - \vec{f}(1,0)$ ← exact change in f

On approximating $\Delta\vec{f}$ by $d\vec{f} = J_{\vec{f}}(1,0) \cdot d\vec{x} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$

$$= \begin{bmatrix} -0.1 \\ -0.3 \end{bmatrix}$$

$$\Rightarrow \quad \vec{f}(0.9, 0.1) - \vec{f}(1,0) \approx \begin{bmatrix} -0.1 \\ -0.3 \end{bmatrix}$$

$$\Rightarrow \quad \vec{f}(0.9, 0.1) = \vec{f}(1,0) - \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 1.7 \end{bmatrix} //.$$

<u>Remark</u>: $\Delta\vec{f} = \begin{bmatrix} -0.1159 \\ -0.2898 \end{bmatrix}$ whereas $d\vec{f} = \begin{bmatrix} -0.1 \\ -0.3 \end{bmatrix}$

↑ by straight forward computation

Therefore, $\Delta\vec{f}$ and $d\vec{f}$ are pretty close //.